

ASYMPTOTICS FOR MOMENTS OF CERTAIN COTANGENT SUMS

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ABSTRACT. In this paper we improve a result on the order of magnitude of certain cotangent sums associated to the Estermann and the Riemann zeta functions.

1. INTRODUCTION

The authors in joint work [7] and the second author in his thesis [10], investigated the distribution of cotangent sums

$$c_0\left(\frac{r}{b}\right) = - \sum_{m=1}^{b-1} \frac{m}{b} \cot\left(\frac{\pi mr}{b}\right)$$

as r ranges over the set

$$\{r : (r, b) = 1, A_0 b \leq r \leq A_1 b\},$$

where A_0, A_1 are fixed with $1/2 < A_0 < A_1 < 1$ and b tends to infinity. They could show that

$$H_k = \int_0^1 \left(\frac{g(x)}{\pi}\right)^{2k} dx,$$

where

$$g(x) = \sum_{l \geq 1} \frac{1 - 2\{lx\}}{l},$$

a function that has been investigated by de la Bretèche and Tenenbaum [5], as well as Balazard and Martin [2, 3]. Bettin [4] could replace the interval $(1/2, 1)$ for A_0, A_1 by the interval $(0, 1)$.

In [8], Theorem 1.1 the authors could determine the order of magnitude of H_k . There are constants $c_1, c_2 > 0$, such that

$$(1.1) \quad c_1 \Gamma(2k+1) \leq \int_0^1 g(x)^{2k} dx \leq c_2 \Gamma(2k+1),$$

for all $k \in \mathbb{N}$, where $\Gamma(\cdot)$ stands for the Gamma function.

In this paper we extend the result of (1.1) to an asymptotic formula valid for arbitrary natural exponents.

Theorem 1.1. *Let*

$$A = \int_0^\infty \frac{\{t\}^2}{t^2} dt$$

and $K \in \mathbb{N}$. There is an absolute constant $C > 0$, such that

$$\int_0^1 |g(x)|^K dx = 2e^{-A}\Gamma(K+1)(1 + O(\exp(-CK))),$$

for $K \rightarrow \infty$.

2. OVERVIEW AND PRELIMINARY RESULTS

Like in the proof of (1.1), a crucial role is played by the relation of $g(x)$ to Wilton's function, established by Balazard and Martin [3] and results about operators related to continued fraction expansions due to Marmi, Moussa and Yoccoz [9].

We recall some fundamental definitions and results.

Definition 2.1. Let $X = (0, 1) \setminus \mathbb{Q}$. Let $\alpha(x) = \{1/x\}$ for $x \in X$. The iterates α_k of α are defined by $\alpha_0(x) = x$ and

$$\alpha_k(x) = \alpha(\alpha_{k-1}(x)), \text{ for } k > 1.$$

Lemma 2.2. Let $x \in X$ and let

$$x = [a_0(x); a_1(x), \dots, a_k(x), \dots]$$

be the continued fraction expansion of x . We define the partial quotient of $p_k(x)$, $q_k(x)$:

$$\frac{p_k(x)}{q_k(x)} = [a_0(x); a_1(x), \dots, a_k(x)], \text{ where, } (p_k(x), q_k(x)) = 1.$$

Then we have

$$a_k(x) = \left\lfloor \frac{1}{\alpha_{k-1}(x)} \right\rfloor,$$

$$p_{k+1} = a_{k+1}p_k + p_{k-1}$$

and

$$q_{k+1} = a_{k+1}q_k + q_{k-1}.$$

Proof. This is Lemma 2.2 of [8]. □

Definition 2.3. Let $x \in X$. Let also

$$\beta_k(x) = \alpha_0(x)\alpha_1(x) \cdots \alpha_k(x), \quad \beta_{-1}(x) = +1$$

$$\gamma_k(x) = \beta_{k-1}(x) \log \frac{1}{\alpha_k(x)}, \text{ where } k \geq 0,$$

so that $\gamma_0(x) = \log(1/x)$.

The number x is called a **Wilton number** if the series

$$\sum_{k \geq 0} (-1)^k \gamma_k(x)$$

converges.

Wilton's function $\mathcal{W}(x)$ is defined by

$$\mathcal{W}(x) = \sum_{k \geq 0} (-1)^k \gamma_k(x)$$

for each Wilton number $x \in (0, 1)$.

Lemma 2.4. *A number $x \in X$ is a Wilton number if and only if $\alpha(x)$ is a Wilton number. In this case we have:*

$$\mathcal{W}(x) = \log \frac{1}{x} - x\mathcal{W}(\alpha(x)).$$

Proof. This is Lemma 2.4 of [8]. □

Definition 2.5. *Let $p > 1$ and $T : L^p \rightarrow L^p$ be defined by*

$$Tf(x) = xf(\alpha(x)).$$

The measure m is defined by

$$m(\mathcal{E}) = \frac{1}{\log 2} \int_{\mathcal{E}} \frac{dx}{1+x},$$

where f is any measurable subset of $(0, 1)$.

Lemma 2.6. *Let $p > 1$, $n \in \mathbb{N}$.*

(i) The measure m is invariant with respect to the map α , i.e.

$$m(\alpha(\mathcal{E})) = m(\mathcal{E}),$$

for all measurable subsets of $\mathcal{E} \subset (0, 1)$.

(ii) For $f \in L^p$ we have

$$\int_0^1 |T^n f(x)|^p dm(x) \leq g^{(n-1)p} \int_0^1 |f(x)|^p dm(x),$$

where

$$g := \frac{\sqrt{5}-1}{2} < 1.$$

Proof. This is Lemma 2.8 of [8]. □

Lemma 2.7. *There is a bounded function $H : (0, 1) \rightarrow \mathbb{R}$, which is continuous in every irrational number, such that*

$$g(x) = \mathcal{W}(x) + H(x).$$

Proof. See Lemma 2.5 of [8]. □

Lemma 2.5 of [8] is based on [3]. In the proof of (1.1) we only use the boundedness of H .

The key to the improvement of (1.1) is the use of more subtle properties of H . We recall the following definitions and results from [3].

Definition 2.8. *For $\lambda \geq 0$, we set*

$$A(\lambda) := \int_0^\infty \{t\}\{\lambda t\} \frac{dt}{t^2},$$

$$F(x) := \frac{x+1}{2}A(1) - A(x) - \frac{x}{2}\log x,$$

$$G(x) := \sum_{j \geq 0} (-1)^j \beta_{j-1}(x) F(\alpha_j(x)),$$

$$B_1(t) := t - [t] - 1/2, \text{ the first Bernoulli function,}$$

$$B_2(t) := \{t\}^2 - \{t\} + 1/6, \text{ } (t \in \mathbb{R}) \text{ the second Bernoulli function.}$$

For $\lambda \in \mathbb{R}$, let

$$\phi_2(\lambda) := \sum_{n \geq 1} \frac{B_2(n\lambda)}{n^2}.$$

Lemma 2.9. *It holds*

$$A(\lambda) = \frac{\lambda}{2} \log \frac{1}{\lambda} + \frac{1 + A(1)}{2} \lambda + O(\lambda^2), \quad \text{as } \lambda \rightarrow 0.$$

Proof. By [3], Proposition 31, formula (74), we have:

$$A(\lambda) = \frac{\lambda}{2} \log \frac{1}{\lambda} + \frac{1 + A(1)}{2} \lambda + \frac{\lambda^2}{2} \phi_2 \left(\frac{1}{\lambda} \right) - \int_{1/\lambda}^{\infty} \phi_2(t) \frac{dt}{t^3}.$$

From Definition 2.8, it follows that $\phi_2(t)$ is bounded. Therefore

$$\frac{\lambda^2}{2} \phi_2 \left(\frac{1}{\lambda} \right) = O(\lambda^2)$$

and

$$\int_{1/\lambda}^{\infty} \phi_2(t) \frac{dt}{t^3} = O(\lambda^2).$$

□

Lemma 2.10. *We have*

$$H(x) = 2 \sum_{j \geq 0} (-1)^{j-1} \beta_{j-1}(x) F(\alpha_j(x)).$$

Proof. In [3] the function Φ_1 is defined by

$$(2.1) \quad \Phi_1(t) := \sum_{n \geq 1} \frac{B_1(nt)}{n} = \sum_{n \geq 1} \frac{\{nt\} - 1/2}{n}.$$

Thus we have

$$(2.2) \quad g(x) = -2\Phi_1(x).$$

By Proposition (2) of [3] we obtain

$$(2.3) \quad \Phi_1(x) = -\frac{1}{2} \mathcal{W}(x) + G(x)$$

almost everywhere.

The proof of Lemma 2.10 follows now from Lemma 2.7, (2.1), (2.2) and (2.3) by the choice

$$(2.4) \quad H = -2G.$$

□

3. PROOF OF THEOREM 1.1

Definition 3.1. *Let $d, h \in \mathbb{N}_0$, $h \geq 1$, $u, v \in (0, \infty)$. Then we define*

$$\mathcal{J}(d, h, u, v) := \{x \in X : T^d l(x) \geq u \text{ and } T^{d+h} l(x) \geq v\}.$$

Lemma 3.2. *We have*

$$m(\mathcal{J}(d, h, u, v)) \leq 2 \exp \left(-2^{\frac{h-2}{2}} v \exp \left(2^{\frac{d-2}{2}} u \right) \right)$$

Proof. This is Lemma 2.13 of [8].

□

Definition 3.3. For $n \in \mathbb{N}$, $x \in X$, we define

$$\mathcal{L}(x, n) := \sum_{v=0}^n (-1)^v (T^v l)(x),$$

where $l(x) = \log(1/x)$.

Definition 3.4. (Definition 2.14 of [8])

We set $j_0 := L - \lfloor \frac{L}{100} \rfloor$, $C_2 := 1/400$. For $j \in \mathbb{Z}$, $j \leq j_0$, we define the intervals

$$I(L, j) := (\exp(-L + j - 1), \exp(-L + j)).$$

For $v \in \mathbb{N}_0$, we set

$$a(L, v) := \exp(-C_2 L + v).$$

$$\mathcal{T}(L, j, 0) := \{x \in I(L, j) \cap X : |\mathcal{L}(x, n) - l(x)| \leq \exp(-C_2 L)\},$$

and for $v \in \mathbb{N}$, we set

$$\mathcal{T}(L, j, v) := \{x \in I(L, j) \cap X : a(L, v - 1) \leq |\mathcal{L}(x, n) - l(x)| \leq a(L, v)\}.$$

For $v, h \in \mathbb{Z}$, $v \geq 1$, $h \geq 0$, we set

$$U(L, j, v, h) := \{x \in \mathcal{T}(L, j, v) : T^h l(x) \geq 2^{-h} a(L, v - 1)\}.$$

Lemma 3.5. There are constants $C_3, C_4 > 0$, such that for $v \geq 1$, we have

$$m(\mathcal{T}(L, j, v)) \leq C_3 \exp\left(-C_4 \exp\left(-C_2 L + v - 1 + \frac{1}{2}(L - j)\right)\right).$$

Proof. This is lemma 2.15 of [8]. □

Definition 3.6. (Definition 2.16 of [8])

We set

$$x_0 := \exp\left(-\left\lfloor \frac{L}{100} \right\rfloor\right).$$

Lemma 3.7. Let $L \in \mathbb{N}$, then

(i)

$$\int_0^1 l(x)^L dx = \Gamma(L + 1)$$

(ii) There is a constant $C_5 > 0$, such that

$$\int_{x_0}^1 l(x)^L dx = O(\Gamma(L + 1) \exp(-C_5 L)).$$

Proof. This is parts (i) and (ii) of Lemma 2.17 of [8]. □

Lemma 3.8. Let $1 < p \leq 2$, such that $pL \in \mathbb{N}$. There is $n_0 \in \mathbb{N}$ and a constant $C_6 > 0$, such that for $n \geq n_0$, we have:

$$\int_0^{x_0} |\mathcal{L}(x, n)^L - l(x)^L|^p dm(x) \leq \Gamma(pL + 1) \exp(-C_6 L).$$

Proof. We write

$$(3.1) \quad \mathcal{L}(x, n) := l(x)(1 + R(x, n)).$$

Let $j \leq j_0$. Then by Definition 3.4, for $x \in \mathcal{T}(L, j, v)$ we have $l(x) \geq L - j$ and therefore we get

$$(3.2) \quad l(x) \geq \frac{L}{200}.$$

By Definition 3.4 we also have

$$(3.3) \quad |\mathcal{L}(x, n) - l(x)| \leq \exp(-C_2 L + v) .$$

From (3.2) and (3.3) we have:

$$(3.4) \quad |R(x, n)| \leq \frac{200}{L} \exp(-C_2 L + v) .$$

We distinguish two cases:

Case 1: Let $v = 0$.

From (3.4) we have

$$(3.5) \quad |R(x, n)| \leq \exp\left(-\frac{C_2}{2}L\right) .$$

$$(3.6) \quad \int_{\mathcal{T}(L, j, 0)} |\mathcal{L}(x, n)^L - l(x)^L|^p dx \leq \int_{\mathcal{T}(L, j, 0)} l(x)^{pL} |(1 + R(x, n))^L - 1|^p dx$$

From (3.5) and (3.6) we have:

$$(3.7) \quad \int_{\mathcal{T}(L, j, 0)} |\mathcal{L}(x, n)^L - l(x)^L|^p dx \leq \exp\left(-\frac{C_2}{3}L\right) \int_{\mathcal{T}(L, j, 0)} l(x)^{pL} dx .$$

Case 2: Let $v \geq 1$.

Because of the fact that

$$L - j \geq \frac{L}{100} ,$$

we have for an appropriate constant $C_7 > 0$ that

$$\max_{x \in I(L, j)} l(x)^L \leq C_7 \min_{x \in I(L, j)} l(x)^L$$

and therefore from (3.4), it follows that

$$(3.8) \quad \begin{aligned} \int_{\mathcal{T}(L, j, v)} |\mathcal{L}(x, n)^L - l(x)^L|^p dx &\leq \exp(-C_2 L + v) m(\mathcal{T}(L, j, v)) \max_{x \in I(L, j)} l(x)^{pL} \\ &\leq C_3 C_7 \exp\left(-C_4 \exp\left(-C_2 L + v - 1 + \frac{1}{2}(L - j)\right)\right) \exp(-C_2 L + v) \min_{x \in I(L, j)} l(x)^{pL} . \end{aligned}$$

From (3.7) and (3.8), we obtain for $j \leq j_0$, the following

$$(3.9) \quad \int_{I(L, j) \cap X} |\mathcal{L}(x, n)^L - l(x)^L|^p dx \leq \exp\left(-\frac{C_2}{3}L\right) \int_{I(L, j)} l(x)^{pL} dx$$

The result of Lemma 3.8 now follows from Lemma 3.7 by summing (3.9) for $j \leq j_0$. \square

Lemma 3.9. *Let $1 < p \leq 2$ and $pL \in \mathbb{N}$. There is a constant $C_8 > 0$, such that*

$$\int_{x_0}^{1/2} |\mathcal{L}(x, n)|^{pL} dx \leq \Gamma(pL + 1) \exp(-C_8 L) .$$

Proof. Lemma 3.9 follows if we apply Lemma 2.22 from [8] with pL instead of L . \square

Lemma 3.10. *Let $0 < \alpha < 1$. Then, there is a constant $C = C(\alpha) > 0$, such that*

$$\int_0^{1/2} x^\alpha l(x)^L dx \leq \Gamma(L + 1) \exp(-CL) ,$$

for all $L \in \mathbb{N}$.

Proof. We have

$$\int_0^{1/2} x^\alpha l(x)^L dx \leq \sum_{0 \leq j \leq j_0} \int_{I(L,j)} x^\alpha l(x)^L dx + \int_{x_0}^{1/2} x^\alpha l(x)^L dx .$$

For $x \in I(L, j) = (\exp(-L + j - 1), \exp(-L + j))$ we have $l(x) \leq L - (j - 1)$ and therefore

$$l(x)^L = O(L^L e^{-j}).$$

Therefore, by Stirling's formula

$$\begin{aligned} \int_{I(L,j)} x^\alpha l(x)^L dx &= O(L^L \exp((\alpha + 1)(-L + j) - j)) \\ &= O(\Gamma(L + 1) \exp(-\alpha L + (\alpha - 1)j + \epsilon L)), \end{aligned}$$

for all $\epsilon > 0$, which proves Lemma 3.10. \square

Lemma 3.11. For $m \in \mathbb{N}_0$, $x \in X$, we have

$$\alpha_m(x) \alpha_{m+1}(x) \leq \frac{1}{2} .$$

Proof. This is Lemma 2.11 of [8]. \square

Definition 3.12. For $l_1, l_2 \in \mathbb{N}_0$, $0 \leq l_1 + l_2 \leq K$, we set

$$\int_{(l_1, l_2)} := \int_0^{1/2} \mathcal{L}(x, n)^{K-l_1-l_2} H(x)^{l_1} ((-1)^{n+1} T^{n+1} \mathcal{W}(x))^{l_2} dx .$$

Lemma 3.13. There is a constant $C_9 > 0$, such that

$$\int_0^{1/2} |g(x)^K - |g(x)||^K dx \leq \Gamma(K + 1) \exp(-C_9 K) .$$

Proof. Let

$$x \in I(K, j) = (\exp(-K + j - 1), \exp(-K + j)) .$$

Let

$$\mathcal{Y}(K, j) = \{x \in I(K, j) : g(x) \leq 0\} .$$

For $x \in \mathcal{Y}(K, j)$ we must have

$$(3.10) \quad x \in \mathcal{T}(K, j, v) \text{ for } v \geq C_2 K \text{ or } |T^n \mathcal{W}(x)| \geq K - j - H ,$$

where

$$H = \sup_{x \in [0, 1]} |H(x)| .$$

For $w \in \mathbb{N}$, let

$$(3.11) \quad \mathcal{V}(K, j, w, n) = \{x \in I(L, j) : L - j - H + w \leq |T^n \mathcal{W}(x)| \leq L - j - H + w + 1\} .$$

Let

$$(3.12) \quad \mathcal{Z}(K, j, w, n) = \mathcal{T}(K, j, v) \cap \mathcal{V}(K, j, w, n) .$$

By Lemma 2.6 (ii) we have:

$$m(\mathcal{V}(K, j, w, n))(K - j - H + w)^2 \leq \int_{\mathcal{V}(L, j, w)} |T^n \mathcal{W}(x)|^2 dm(x) \leq g^{2(n-1)} \int_0^1 |\mathcal{W}(x)|^2 dm(x) .$$

Thus

$$(3.13) \quad m(\mathcal{V}(K, j, w, n)) \leq g^{2(n-1)} \int_0^1 |\mathcal{W}(x)|^2 dm(x) (L - j - H + w)^{-2} .$$

We have

$$(3.14) \quad |g(x)^K - |g(x)||^K| \leq 2|g(x)|^K$$

and for $x \in \mathcal{Z}(K, j, w, n)$

$$(3.15) \quad |g(x)| \leq b(x, K, j, n) + |\mathcal{L}(x, n) - l(x)|,$$

where $b(x, K, j, n) := l(x) + L - j + w + 1$. Thus, from (3.14) we get

$$(3.16) \quad \int_{\mathcal{Z}(K, j, w, n)} |g(x)^K - |g(x)||^K| dx \leq 2^K \left(\sup_{x \in I(K, j)} |b(x, K, j, n)|^K + \int_{I(K, j)} |\mathcal{L}(x, n) - l(x)|^K dx \right) \times (m(\mathcal{T}(K, j, v)) + m(\mathcal{V}(K, j, w, n))) .$$

From Lemma 3.5, Lemma 3.8, (3.15), (3.16) we get by summation over j, v and w :

$$(3.17) \quad \int_0^{x_0} |g(x)^K - |g(x)||^K| dx \leq \Gamma(K+1) \exp(-C_{10}K),$$

where $x_0 := x_0(K) = \exp(-\lfloor \frac{K}{100} \rfloor)$. From Lemma 3.5, we obtain:

$$(3.18) \quad \int_{x_0}^{1/2} |g(x)^K - |g(x)||^K| dx \leq \Gamma(K+1) \exp(-C_{11}K) .$$

Lemma 3.13 now follows from (3.17) and (3.18). \square

Lemma 3.14. *We have*

$$\int_0^{1/2} g(x)^K dx = \sum_{\substack{(l_1, l_2) \in \mathbb{N}_0^2 \\ 0 \leq l_1 + l_2 \leq K}} \frac{K!}{(K - l_1 - l_2)! l_1! l_2!} \int_{(l_1, l_2)} .$$

Proof. From formula (3) of [8] we have:

$$\mathcal{W}(x) = \mathcal{L}(x, n) + (-1)^{n+1} T^{n+1} \mathcal{W}(x) .$$

By Lemma 2.7, we obtain

$$g(x) = \mathcal{L}(x, n) + H(x) + (-1)^{n+1} T^{n+1} \mathcal{W}(x) .$$

Lemma 3.14 now follows by the Multinomial Theorem. \square

Definition 3.15. *For (l_1, l_2) as in Definition 3.12 we set*

$$\begin{aligned} \int_{(l_1, l_2)}^{(1)} &:= \int_0^{1/2} l(x)^{K-l_1-l_2} H(x)^{l_1} [(-1)^{n+1} T^{n+1} l(x)]^{l_2} dx \\ \int_{(l_1, l_2)}^{(2)} &:= \int_0^{1/2} (\mathcal{L}(x, n)^{K-l_1-l_2} - l(x)^{K-l_1-l_2}) H(x)^{l_1} [(-1)^{n+1} T^{n+1} \mathcal{W}(x)]^{l_2} dx . \end{aligned}$$

Lemma 3.16.

$$\int_{(l_1, l_2)} = \int_{(l_1, l_2)}^{(1)} + \int_{(l_1, l_2)}^{(2)} .$$

Proof. Obvious. \square

We now show, that the integrals $\int_{(l_1, l_2)}^{(2)}$ for all l_1, l_2 and $\int_{(l_1, l_2)}^{(1)}$, if $l_2 > 0$ are negligible.

Lemma 3.17. *There is an $n_0 = n_0(K) \in \mathbb{N}$, such that for $n \geq n_0$ we have for $i = 1, 2$ and all $l_1 \leq K$ and $l_2 > 0$ the following*

$$\int_{(l_1, l_2)}^{(i)} \leq (K(2K)!)^{-1}.$$

Proof. We choose $1 < p \leq 2$. We set $L = K - l_1 - l_2$ and apply Lemma 3.8 with $p = 2$ to obtain from the inequality of Cauchy-Schwarz:

$$\int_{(l_1, l_2)}^{(i)} \leq \left(\int_0^{1/2} I(x)^2 dx \right)^{1/2} \left(\int_0^{1/2} |T^{n+1} \mathcal{W}(x)|^{2l_2} dx \right)^{1/2} \sup_{x \in [0, 1/2]} |H(x)^{l_2}|,$$

where

$$I(x) := l(x)^L, \text{ for } i = 1$$

and

$$I(x) := \mathcal{L}(x, n)^L - l(x)^L, \text{ for } i = 2.$$

By Lemma 2.6 we obtain the result if we choose n_0 sufficiently large. \square

Lemma 3.18. *Assume L_0 is sufficiently large and that $L := K - l_1 \geq L_0$. There are constants $C_9, C_{10} > 0$, such that*

$$\left| \int_{(l_1, 0)}^{(2)} \right| \leq C_0^{l_1} K^{l_1} \Gamma(K + 1 - l_1) \exp(-C_{13}K).$$

Proof. Let $|H(x)| \leq C_{11}$ with $C_{11} > 0$. We choose p , $1 < p \leq 2$, such that $pL \in \mathbb{N}$. We define $\epsilon > 0$ by $(1 - \epsilon)^{-1} = p$. Then by Lemma 3.8 and Hölder's inequality we have

$$\begin{aligned} \int_{(l_1, 0)}^{(2)} &\leq \left(\int_0^{1/2} |\mathcal{L}(x, n)^L - l(x)^L|^p dx \right)^{1/p} \left(\int_0^{1/2} |H(x)|^{l_1/\epsilon} dx \right)^\epsilon \\ &\leq \Gamma(pL + 1)^{1/p} \exp\left(-\frac{C_6}{p}L\right) C_{11}^{l_1}. \end{aligned}$$

By Stirling's formula

$$\int_{(l_1, l_2)}^{(2)} \leq (pL)^L \exp\left(-\frac{L - 3\epsilon}{p}\right) \exp\left(-\frac{C_6}{p}L\right)$$

for sufficiently large L .

Since $\epsilon \rightarrow 0$ for $L \rightarrow \infty$, the result of Lemma 3.18 follows. \square

Lemma 3.19. *There is a constant $C_{15} > 0$, such that*

$$\int_0^{1/2} g(x)^K dx = \sum_{0 \leq l_1 \leq K} \binom{K}{l_1} \int_{(l_1, 0)}^{(1)} + O(\Gamma(K + 1) \exp(-C_{15}K)).$$

Proof. This follows from Lemmas 3.16 - 3.18. \square

Definition 3.20. *Let $0 \leq m \leq l_1$. Then we set*

$$\int^{(l_1, m)} := \int_0^{1/2} l(x)^{2k-l_1} (-2F(x))^{l_1-m} \left(\sum_{j \geq 0} (-1)^{j-1} \beta_{j-1} F(\alpha_j(x)) \right)^m dx.$$

Lemma 3.21.

$$\int_{(l_1,0)}^{(1)} = \sum_{m=0}^{l_1} \binom{l_1}{m} \int^{(l_1,m)}.$$

Proof. This follows from Lemma 3.11, Definition 3.15, 3.20 and the Binomial Theorem. \square

Lemma 3.22. *There is a constant $C_{13} > 0$, such that*

$$\int_0^{1/2} g(x)^K dx = \sum_{0 \leq l_1 \leq K} \binom{K}{l_1} \int^{(l_1,0)} + O(\Gamma(K+1) \exp(-C_{13}K)).$$

Proof. Let $m > 0$. We have

$$\beta_{j-1} = x\alpha_1(x) \cdots \alpha_{j-1}(x).$$

By Lemma 3.11 we have for an absolute constant C_{14} the following

$$\left| \sum_{j>0} (-1)^{j-1} \beta_{j-1} F(\alpha_j(x)) \right| < C_{14}x, \text{ if } x \in (0,1).$$

We also have

$$|-2F(x)| < C_{15}.$$

By Lemma 3.10, we therefore have

$$\begin{aligned} \left| \int^{(l_1,m)} \right| &\leq C_{15}^{l_1-m} \left| \int_0^{1/2} l(x)^{K-l_1} \left(\sum_{j>0} (-1)^{j-1} \beta_{j-1} F(\alpha_j(x)) \right)^m dx \right| \\ &\leq \Gamma(K-l_1+1) (3C_{14}C_{15})^{l_1}. \end{aligned}$$

Lemma 3.22 follows by summation over l_1 . \square

Definition 3.23. *For $0 \leq l_1 \leq K$ we set*

$$Int(l_1) := \int_0^{1/2} l(x)^{K-l_1} (-2F(x))^{l_1} dx.$$

For $0 \leq m \leq l_1$ we set

$$Int(l_1, m) := \int_0^{1/2} l(x)^{K-l_1} (-A(1))^{l_1-m} R(x)^m dx,$$

where

$$R(x) := -xA(1) + A(x) + \frac{x}{2} \log x.$$

Lemma 3.24. *We have*

$$Int(l_1) = \sum_{m=0}^{l_1} \binom{l_1}{m} Int(l_1, m).$$

Proof. This follows by Definition 3.23 and the Binomial Theorem. \square

Lemma 3.25. *There is a constant $C_{16} > 0$, such that*

$$\int_0^{1/2} g(x)^K dx = \sum_{0 \leq l_1 \leq K} \binom{K}{l_1} \int_0^{1/2} l(x)^{K-l_1} (-A(1))^{l_1} dx + O(\Gamma(K+1) \exp(-C_{16}K)).$$

Proof. This follows in a similar manner as the result of Lemma 3.22 by application of Lemma 3.10 and summation over l_1 . \square

4. CONCLUSION OF THE PROOF OF THEOREM 1.1

We have

$$\binom{K}{l_1} \int_0^1 l(x)^{K-l_1} dx = \binom{K}{l_1} \Gamma(K-l_1+1) = \frac{1}{l_1!} \Gamma(K+1).$$

From Lemmas 3.7 and 3.25, we therefore get

$$(4.1) \quad \int_0^{1/2} g(x)^K dx = \left(\sum_{l_1=0}^{\infty} \frac{1}{l_1!} (-A(1))^{l_1} \right) \Gamma(K+1) + O(\Gamma(K+1) \exp(-C_{18}K)).$$

From Lemma 3.13 and (4.1) we obtain

$$(4.2) \quad \int_0^{1/2} |g(x)|^K dx = \left(\sum_{l_1=0}^{\infty} \frac{1}{l_1!} (-A(1))^{l_1} \right) \Gamma(K+1) + O(\Gamma(K+1) \exp(-C_{19}K)).$$

Since

$$\int_0^{1/2} |g(x)|^K dx = \int_{1/2}^1 |g(x)|^K dx,$$

this concludes the proof of Theorem 1.1.

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